

# NONUNIQUENESS OF SEMIDIRECT DECOMPOSITIONS FOR SEMIDIRECT PRODUCTS WITH DIRECTLY DECOMPOSABLE FACTORS AND APPLICATIONS FOR DIHEDRAL GROUPS

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**Abstract.** Nonuniqueness of semidirect decompositions of groups is an insufficiently studied question in contrast to direct decompositions. We obtain some results about semidirect decompositions for semidirect products with factors which are nontrivial direct products. We deal with a special case of semidirect product when the twisting homomorphism acts diagonally on a direct product, as well as for the case when the extending group is a direct product. We give applications of these results in the case of generalized dihedral groups and classic dihedral groups  $D_{2n}$ .

**Key words.** semidirect product, direct product, diagonal action, generalized dihedral group

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## 1. Introduction.

**1.1. Background.** The aim of this article is to study semidirect decompositions of groups both in general and special cases.

By the well known Krull-Remak-Schmidt theorem the multiset of isomorphism types of indecomposable direct factors for groups satisfying ascending and descending chain conditions on normal subgroups does not depend on the order of factors. Thus direct decompositions of such groups, e.g. finite groups, may be considered understood.

Few results of this type are known for semidirect and Zappa-Szep decompositions. One can mention the Schur-Zassenhaus theorem as an example.

We consider cases when the base group or the extending group is a direct product. We present a general result which allows to characterize some semidirect decompositions in the case when the base group is a direct product and the twisting homomorphism acts diagonally, Prop.2.4. We also show that any semidirect product is isomorphic to a semidirect product with injective twisting homomorphism acting diagonally, Prop.2.6. Finally we obtain a nonuniqueness result of semidirect decomposition in the case when the extending group is a direct product, Prop.2.8. We give

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applications of some of these results in the case which is relatively easy to understand - finite dihedral groups, both classic and generalized.

We use traditional multiplicative notation for general groups and additive notation for abelian groups. In this article the dihedral group of order  $m$  is denoted by  $D_m$ . The cyclic group of order  $m$  is denoted by  $\mathbb{Z}_m$ , in additive notation we assume that  $\mathbb{Z}_m = \langle 1 \rangle$  and its elements are residues mod  $m$ . For any  $m|n$  we usually identify  $\mathbb{Z}_m$  with the corresponding subgroup of  $\mathbb{Z}_n$ .

## 1.2. Basic facts about semidirect products.

We remind the reader that an external semidirect product of groups  $N$  (*base group*) and  $K$  (*extending group*) is the group  $N \rtimes_{\varphi} H = (N \times H, \cdot)$  where the group product is defined on the Cartesian product  $N \times H$  using a group homomorphism (*twisting homomorphism*)  $\varphi \in \text{Hom}(H, \text{Aut}(N))$  as follows:  $(n_1, h_1) \cdot (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1 h_2)$ . Sets  $\tilde{N} = N \times \{e_H\}$  and  $\tilde{H} = \{e_N\} \times H$  are subgroups in  $N \times H$ .

A group  $G$  is an internal semidirect product of its subgroups  $N$  and  $H$  if  $N$  is a normal subgroup,  $G = NH$  and  $N \cap H = \{e\}$ . If a group  $G$  is finite then for  $G$  to be an internal semidirect product  $NH$  is equivalent to 1)  $N$  being normal in  $G$ , 2)  $|H| \cdot |K| = |G|$  and 3)  $N \cap K = \{e\}$ . In the internal case the twisting homomorphism  $H \rightarrow \text{Aut}(N)$  is given by the map  $h \mapsto (n \rightarrow hnh^{-1})$ , for any  $n \in N$ ,  $h \in H$ .

Both expressions will be called semidirect decompositions of  $G$ . If the twisting homomorphism is not discussed, we omit it and use the notation  $\rtimes$ .

We consider direct product to be a special case of semidirect product with the twisting homomorphism being trivial.

For relevant treatment see [4], [5].

## 2. Main results.

### 2.1. Diagonal semidirect products.

**2.1.1. Automorphisms of direct products.** We introduce a linear algebra style notation for direct products of groups.

Let  $G = G_1 \times G_2$ . Encode the element  $(g_1, g_2)$  as a column  $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ . If  $\varphi \in \text{Aut}(G)$ , then  $\varphi\left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right) = \begin{bmatrix} \varphi_1(g_1, g_2) \\ \varphi_2(g_1, g_2) \end{bmatrix}$ . One can check, that for all relevant parameter values  $\varphi_i$  satisfy the following properties:

1.  $\varphi_i(ab, e) = \varphi_i(a, e)\varphi_i(b, e)$ ,
2.  $\varphi_i(e, ab) = \varphi_i(e, a)\varphi_i(e, b)$ ,
3.  $\varphi_i(a, b) = \varphi_i(a, e)\varphi_i(e, b) = \varphi_i(e, b)\varphi_i(a, e)$ ,

Define  $\varphi_{11}(g_1) = \varphi_1(g_1, e)$ ,  $\varphi_{12}(g_2) = \varphi_1(e, g_2)$ ,  $\varphi_{21}(g_1) = \varphi_2(g_1, e)$ ,  $\varphi_{22}(g_2) = \varphi_2(e, g_2)$ , for all  $g_i \in G_i$ . All functions  $\varphi_{ij}$  are group homomorphisms. Thus  $\varphi_i(g_1, g_2) = \varphi_i(g_1, e)\varphi_i(e, g_2) = \varphi_{i1}(g_1)\varphi_{i2}(g_2)$ .

We can encode action of  $\varphi$  as follows:  $\varphi\left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right) = \left[\begin{array}{c|c} \varphi_{11}(g_1) & \varphi_{12}(g_2) \\ \hline \varphi_{21}(g_1) & \varphi_{22}(g_2) \end{array}\right]$ . Thus an automorphism  $\varphi \in \text{Aut}(G_1 \times G_2)$  is determined by 4 group homomorphisms  $\varphi_{ij} : G_j \rightarrow G_i$ .

**DEFINITION 2.1.** We call  $\varphi \in \text{Aut}(G_1 \times G_2)$ ,  $G_1 \neq \{e\}$ ,  $G_2 \neq \{e\}$ , a diagonal automorphism if  $\varphi_{12}$  and  $\varphi_{21}$  are trivial homomorphisms.

**DEFINITION 2.2.** We call  $(G_1 \times G_2) \rtimes_{\varphi} H$  a diagonal semidirect product if  $\varphi(h)$  is a diagonal  $G_1 \times G_2$ -automorphism for any  $h \in H$ . Explicitly, there are group homomorphisms  $\varphi_{ii}(h) : G_i \rightarrow G_i$  such that  $\varphi(h)(g_1, g_2) = (\varphi_{11}(h)(g_1), \varphi_{22}(h)(g_2))$ .

**REMARK 2.3.**  $\text{Ker}(\varphi) = \text{Ker}(\varphi_{11}) \cap \text{Ker}(\varphi_{22})$ . Note that  $G_i$  may not be indecomposable as direct factors. Described encodings and diagonal semidirect products can be generalized to cases when the base groups splits into an arbitrary finite number of direct factors. Similar encodings can be used considering internal semidirect products.

### 2.1.2. Semidirect decompositions of diagonal semidirect products.

We present a proposition showing nonuniqueness of semidirect decomposition for diagonal semidirect products. Vaguely speaking, any direct factor of the base group which is invariant with respect to the initial twisting homomorphism can be moved to the extending group (nonnormal semidirect factor) to enlarge it. The new twisting homomorphism is such that the moved direct factor acts trivially on the remaining part of the base group.

**PROPOSITION 2.4.** Let  $N_1, N_2, H$  be groups. Let  $G = (N_1 \times N_2) \rtimes_{\varphi} H$  be a diagonal semidirect product,  $\varphi(h)(g_1, g_2) = (\varphi_{11}(h)(g_1), \varphi_{22}(h)(g_2))$ . Then the following statements hold.

1.  $G \simeq N_1 \rtimes_{\Phi_{11}} (N_2 \rtimes_{\varphi_{22}} H)$ , for some  $\Phi_{11} \in \text{Hom}(N_2 \rtimes_{\varphi_{22}} H, \text{Aut}(N_2))$ .
2.  $\text{Ker}(\Phi_{11}) = \widetilde{N_2} \text{Ker}(\varphi_{11})$ .
3. If  $\varphi_{11}(h) = \text{id}_{N_1}$ , for any  $h \in H$ , i.e.  $\varphi(h)\left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right) = \left[\begin{array}{c|c} g_1 & e \\ \hline e & \varphi_{22}(h)(g_2) \end{array}\right]$ , then  $G \simeq N_1 \times (N_2 \rtimes_{\varphi_{22}} H)$ .

*Proof.*

1. Consider  $N_1 \rtimes_{\Phi_{11}} (N_2 \rtimes_{\varphi_{22}} H)$  where  $\Phi_{11}(n_2, h) = \varphi_{11}(h)$ . It is directly checked that  $\Phi_{11} \in \text{Hom}(N_2 \rtimes H, \text{Aut}(N_1))$ . We will prove that

$$(N_1 \times N_2) \rtimes_{\varphi} H \simeq N_1 \rtimes_{\Phi_{11}} (N_2 \rtimes_{\varphi_{22}} H).$$

Define a bijective map  $f : (N_1 \times N_2) \rtimes_{\varphi} H \rightarrow N_1 \rtimes_{\Phi_{11}} (N_2 \rtimes_{\varphi_{22}} H)$  by  $f((n_1, n_2), h) = (n_1, (n_2, h))$ , for all  $n_i \in N_i$ ,  $h \in H$ . We prove that  $f$  is a group homomorphism.

Let  $a, a' \in N_1$ ,  $b, b' \in N_2$ ,  $h, h' \in H$ . We have that

$$\begin{aligned} ((a, b), h) \cdot ((a', b'), h') &= ((a, b)\varphi(h)(a', b'), hh') = \\ &= ((a, b)(\varphi_{11}(h)(a'), \varphi_{22}(h)(b')), hh') = ((a\varphi_{11}(h)(a'), b\varphi_{22}(h)(b')), hh'). \end{aligned}$$

On the other hand,

$$\begin{aligned} (a, (b, h)) \cdot (a', (b', h')) &= (a\Phi_{11}(b, h)(a'), (b, h) \cdot (b', h')) = \\ &= (a\varphi_{11}(h)(a'), (b\varphi_{22}(h)(b'), hh')). \end{aligned}$$

We see that  $f$  is a group isomorphism.

2.  $\text{Ker}(\Phi) = \{(n_2, h) | h \in \text{Ker}(\varphi_{11})\} = \widetilde{N_2 \text{Ker}(\varphi_{11})}$ .

3. In notations given above,  $\varphi_{11}(h) = id_{N_1}$  implies  $\Phi_{11}(n_2, h) = id_{N_1}$ , for any  $n_2 \in N_2$ ,  $h \in H$ . Thus it is the direct product.  $\square$

EXAMPLE 2.5. Let  $G = (\mathbb{Z}_7 \times \mathbb{Z}_9) \rtimes_{\varphi} \mathbb{Z}_3$ , where  $\varphi(1)\left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right) = \begin{bmatrix} g_1^2 & e \\ e & g_2^4 \end{bmatrix}$ . In additive notation this can be simplified as follows.  $\varphi(1)\left(\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\right) = \begin{bmatrix} 2g_1 & 0 \\ 0 & 4g_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ .  $G$  can be defined as the subgroup of  $\Sigma_{16}$  generated by three permutations:

- a)  $(1, \dots, 7)$  (generating  $\mathbb{Z}_7$ ),
- b)  $(8, \dots, 16)$  (generating  $\mathbb{Z}_9$ ) and
- c)  $\underbrace{(1, 2, 4)}_{\mathbb{Z}_7} \underbrace{(3, 6, 5)(8, 11, 14)(9, 15, 12)}_{\mathbb{Z}_9}$  (generating action of  $\mathbb{Z}_3$  on  $\mathbb{Z}_7 \times \mathbb{Z}_9$ ).

We have that  $G \simeq \mathbb{Z}_7 \rtimes (\mathbb{Z}_9 \rtimes_4 \mathbb{Z}_3) \simeq \mathbb{Z}_9 \rtimes (\mathbb{Z}_7 \rtimes_2 \mathbb{Z}_3)$ .

**2.1.3. Factoring out extending groups.** We give a proposition showing that any semidirect product can be isomorphic to a diagonal semidirect product with a factored extending group.

PROPOSITION 2.6. *Let  $N, H$  be groups. Let  $\varphi \in \text{Hom}(H, \text{Aut}(N))$  and  $K \leq \text{Ker}(\varphi)$ ,  $K \trianglelefteq H$ . Then  $N \rtimes_{\varphi} H \simeq (N \times K) \rtimes (H/K)$ .*

*Proof.* Choose a right  $K$ -transversal  $T$ . For any  $h \in H$  there are uniquely defined  $k_h \in K$ ,  $t_h \in T$  such that  $h = k_h t_h$ . Thus there is a bijective map  $p : H \rightarrow K \times T$  given by  $p(h) = (p_1(h), p_2(h)) = (k_h, t_h)$ . Define the standard quotient group structure on  $T$  by  $t_i \cdot t_j = p_2(t_i t_j)$ ,  $t_i^{-1} = p_2(t_i^{-1})$ ; clearly  $T \simeq H/K$ ,  $p_2 : H \rightarrow T$  is a group homomorphism.

Define a semidirect product  $(N \times K) \rtimes_{\Phi} T$  with  $\Phi(t)(n, k) = (\varphi(t)(n), tkt^{-1})$ . Note that  $\Phi$  defines a diagonal semidirect product.

Define a bijective map  $f : N \rtimes_{\varphi} H \rightarrow (N \times K) \rtimes_{\Phi} T$  by

$$f(n, h) = ((n, p_1(h)), p_2(h)) = ((n, k_h), t_h) = ((n, ht_h^{-1}), t_h).$$

We prove that  $f$  is a group homomorphism. Let  $n, n' \in N$ ,  $h, h' \in H$ . Consider the product in  $N \rtimes_{\varphi} H$ , we have that  $(n, h) \cdot (n', h') = (n\varphi(h)(n'), hh')$ . Thus  $f((n, h)(n', h')) = f(n\varphi(h)(n'), hh') = (n\varphi(h)(n'), p_1(hh'), p_2(hh'))$ .

Now consider the product in  $(N \times K) \rtimes_{\Phi} T$ .

Suppose that  $f(n, h) = ((n, ht^{-1}), t)$ ,  $f(n', h') = ((n', h't'^{-1}), t')$ . Then

$$\begin{aligned} f(n, h)f(n', h') &= ((n, ht^{-1}), t)((n', t'^{-1}), t') = ((n, ht^{-1})\Phi(t)(n', t'^{-1}), tt') = \\ &= ((n, ht^{-1})(\varphi(t)(n'), th't'^{-1}t^{-1}), tt') = ((n\varphi(t)(n'), ht^{-1}th't'^{-1}t^{-1}), tt') = \\ &= ((n\varphi(h)(n'), hh'(tt')^{-1}), tt'). \end{aligned}$$

Since  $p_2$  is a group homomorphism, we have that  $tt' = p_2(h)p_2(h') = p_2(hh')$  and therefore  $f((n, h)(n', h')) = f(n, h)f(n', h')$ . Thus  $f$  is a group isomorphism.  $\square$

REMARK 2.7. *If  $K = \text{Ker}(\varphi)$ , then  $\Phi$  is injective. Thus Prop.2.6 shows that any semidirect product is isomorphic to a diagonal semidirect product with an injective twisting homomorphism which can be a candidate normal form for semidirect products.*

## 2.2. Directly decomposable extending groups.

We show that a direct factor of the extending group can be transferred to the base group.

PROPOSITION 2.8. *Let  $N, H_1, H_2$  be groups Then*

$$N \rtimes_{\varphi} (H_1 \times H_2) \simeq (N \rtimes_{\varphi_1} H_1) \rtimes_{\varphi_2} H_2,$$

where  $\varphi_1(h_1)(n) = \varphi(h_1, e_{H_2})(n)$  and  $\varphi_2(h_2)(n, h_1) = (\varphi(e_{H_1}, h_2)(n), h_1)$ , for all

$n \in N, h_i \in H_i$ .

*Proof.* It is checked that  $\varphi_i$  are group homomorphisms.

We prove that the map  $f : N \rtimes (H_1 \times H_2) \longrightarrow (N \rtimes H_1) \rtimes H_2$  given by  $f(n, (h_1, h_2)) = ((n, h_1), h_2)$  is a group homomorphism.

Let  $n, n' \in N, h_i, h'_i \in H_i$ .

Consider the product  $(n, (h_1, h_2)) \cdot (n', (h'_1), h'_2)$  in  $N \rtimes_{\varphi} (H_1 \times H_2)$ :

$$(n, (h_1, h_2)) \cdot (n', (h'_1), h'_2) = (n\varphi(h_1, h_2)(n'), (h_1h'_1), h_2h'_2).$$

Consider the product  $((n, h_1), h_2) \cdot ((n', h'_1), h'_2)$  in  $(N \rtimes H_1) \rtimes H_2$ :

$$\begin{aligned} ((n, h_1), h_2) \cdot ((n', h'_1), h'_2) &= (((n, h_1)\varphi_2(h_2)(n', h'_1)), h_2h'_2) = \\ &= (((n, h_1)(\varphi(e, h_2)(n'), h'_1)), h_2h'_2) = ((n\varphi_1(h_1)(\varphi(e, h_2)(n')), h_1h'_1), h_2h'_2) = \\ &= ((n\varphi(h_1, h_2)(n'), h_1h'_1), h_2h'_2). \end{aligned}$$

We see that both products have equal corresponding components and thus  $f$  is a group isomorphism.  $\square$

EXAMPLE 2.9. Let  $G = \mathbb{Z}_7 \rtimes_{\varphi} (\mathbb{Z}_2 \times \mathbb{Z}_3)$  where  $\varphi(x, y)(1) \equiv (-1)^x 2^y \pmod{7}$ .  $G$  can be defined as the subgroup of  $\Sigma_7$  generated by three permutations:

- a)  $(1, \dots, 7)$  (generating  $\mathbb{Z}_7$ ),
- b)  $(1, 6)(2, 5)(3, 4)$  (generating action of  $\mathbb{Z}_2$  on  $\mathbb{Z}_7$ ) and
- c)  $(1, 2, 4)(3, 6, 5)$  (generating action of  $\mathbb{Z}_3$  on  $\mathbb{Z}_7$ ).

Then  $G \simeq D_{2 \cdot 7} \rtimes \mathbb{Z}_3 \simeq (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ .

### 2.3. Applications.

#### 2.3.1. Generalized dihedral groups.

We remind the reader that an external semidirect product  $D(A) = A \rtimes_{\varphi} \mathbb{Z}_2$  is called *generalized dihedral group* provided 1)  $A$  is abelian and 2)  $\varphi(1)(g) = -g$  for any  $g \in A$ , in additive notation. We can also denote  $D(A)$  by  $A \rtimes_{-1} \mathbb{Z}_2$ .

Using the known classification of finite abelian groups we can assume that  $A = \bigoplus_{i=1}^n \mathbb{Z}_{m_i}$ . We use linear algebra style encoding - we encode  $(g_1, \dots, g_n) \in A$  as

a column vector  $\begin{bmatrix} g_1 \\ \dots \\ g_n \end{bmatrix}$ . Notations introduced in section 2.1.1 are modified for

additive group notation. The action of the twisting homomorphism is given by scalar

or matrix multiplication:  $\varphi(1)\left(\begin{bmatrix} g_1 \\ \dots \\ g_n \end{bmatrix}\right) = \left[\begin{array}{c|c|c} (-g_1) & 0 & 0 \\ \hline 0 & \dots & 0 \\ \hline 0 & 0 & (-g_n) \end{array}\right] = -\begin{bmatrix} g_1 \\ \dots \\ g_n \end{bmatrix} =$

$(-\mathbf{E}_n) \cdot \begin{bmatrix} g_1 \\ \dots \\ g_n \end{bmatrix}$ , where  $\mathbf{E}_n$  is the  $n \times n$  identity matrix.

REMARK 2.10. *Generalized dihedral groups are diagonal semidirect products with an injective twisting homomorphism.*

PROPOSITION 2.11. *Let  $A = \bigoplus_{i=1}^n \mathbb{Z}_{m_i}$ , let  $A = A_1 \oplus A_2$ , where  $A_1 = \bigoplus_{i=1}^{n_1} \mathbb{Z}_{m_i}$ ,  $A_2 = \bigoplus_{i=n_1+1}^n \mathbb{Z}_{m_i}$ . Then*

$$D(A) \simeq A_1 \rtimes (A_2 \rtimes_{-1} \mathbb{Z}_2) = A_1 \rtimes D(A_2) \simeq A_1 \rtimes D(A/A_1).$$

*Proof.*  $D(A) = (A_1 \oplus A_2) \rtimes_{\varphi} \mathbb{Z}_2$ , where  $\varphi(1)(g) = -g$ , for any  $g \in A$ . Thus  $\varphi(g_1, g_2) = (-g_1, -g_2)$ , for any  $g_i \in G_i$ . It follows that  $D(A)$  is a diagonal semidirect product with respect to  $A_1 \oplus A_2$  decomposition. According to Proposition 2.4 we have that  $D(A) \simeq A_1 \rtimes_{\Phi_{11}} (A_2 \rtimes_{\varphi_{22}} \mathbb{Z}_2) = A_1 \rtimes D(A_2)$ , where  $\Phi_{11}(g_2, 1)(g_1) = \varphi_{11}(1)(g_1) = -g_1$ .  $\square$

EXAMPLE 2.12. *Let  $G = D(\mathbb{Z}_3 \oplus \mathbb{Z}_5)$ .  $G$  can be defined as a subgroup of  $\Sigma_8$  generated by permutations  $(1, 2, 3)$ ,  $(4, 5, 6, 7, 8)$  and  $(1, 2)(4, 7)(5, 6)$ . Then  $G \simeq \mathbb{Z}_3 \rtimes D_{2 \cdot 5} \simeq \mathbb{Z}_5 \rtimes D_{2 \cdot 3}$ .*

### 2.3.2. Dihedral groups.

Classic dihedral groups are special cases of generalized dihedral groups when the base group is a cyclic group. We give a complete description of semidirect decompositions of  $D_{2n}$  using both Proposition 2.4 and ad hoc computations.

We use a classical presentation of dihedral groups:

$$D_{2n} = \langle a, x \mid a^n = e, x^2 = e, xax = a^{n-1} \rangle = \langle a \rangle \cup \langle a \rangle x.$$

We note that  $D_2 \simeq \mathbb{Z}_2$  and  $D_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ , in all other cases  $D_{2n}$  is nonabelian.

*Subgroups.* Let  $n \in \mathbb{N}$ ,  $n \geq 3$ ,  $d \in \mathbb{N}$ ,  $d \mid n$ ,  $m = \frac{n}{d}$ . It is known that  $D_{2n}$  has the following subgroups, see [2].

1. For each  $m \in \mathbb{N}$  such that  $m \mid n$  there is a subgroup

$$A_m = \langle a^{\frac{n}{m}} \rangle = \langle a^d \rangle = \{e, a^d, a^{2d}, \dots, a^{(m-1)d}\} \simeq \mathbb{Z}_m.$$

$A_m \trianglelefteq D_{2n}$  for all  $m$ . The number of such subgroups is  $d(n)$  (the number of natural  $n$ -divisors).

2. For each  $m \in \mathbb{N}$  such that  $m|n$  and each  $r \in \mathbb{Z}_{\frac{n}{m}} = \mathbb{Z}_d$  there is a subgroup

$$B_{2m,r} = \langle a^{\frac{n}{m}}, a^r x \rangle = \langle a^d, a^r x \rangle = A_m \cup A_m(a^r x) \simeq D_{2m}.$$

The number of such subgroups is  $\sigma(n)$ .

If  $2|n$  then  $B_{n,r} \trianglelefteq D_{2n}$ . In all other cases, if  $1 < m < n$  then  $B_{2m,r} \not\trianglelefteq D_{2n}$ .

*Classical decompositions.* It known that  $D_{2n} \simeq \mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2$  where the twisting homomorphism is  $\varphi(1)(g) = -g$ . In internal terms,  $D_{2n} = A_n \rtimes B_{2,r}$ , for all  $r \in \mathbb{Z}_n$ . If  $2|n$  and  $4 \nmid n$ , then  $D_{2n} \simeq D_n \times \mathbb{Z}_2$ , or, in internal terms,  $D_{2n} = B_{n,r} \times A_2$ . where  $r \in \mathbb{Z}_2$ .

*External semidirect decompositions of  $D_{2n}$ .* Using Proposition 2.4 we get an exhaustive description of external semidirect decompositions of  $D_{2n}$ .

PROPOSITION 2.13.

1.  $D_{2n} \simeq \mathbb{Z}_m \rtimes_{\varphi} D_{\frac{2n}{m}}$ , for any  $m \in \mathbb{N}$ ,  $m|n$ , such that  $GCD(m, \frac{n}{m}) = 1$ .  $\varphi$  is defined as follows: if  $D_{\frac{2n}{m}} = \langle a, x | a^{\frac{2n}{m}} = e, x^2 = e, xax = a^{-1} \rangle$  then  $\varphi(a)(1) = 1$  and  $\varphi(x)(1) = -1$ .
2.  $D_{2n} \simeq D_n \rtimes_{\varphi} \mathbb{Z}_2$ , if  $n = 2^{\alpha}q$ ,  $\alpha \in \mathbb{N}$ .  $\varphi$  is defined as follows: if  $D_n = \langle a, x | a^{\frac{n}{2}} = e, x^2 = e, xax = a^{-1} \rangle$  then  $\varphi(1)(a) = a^{-1}$  and  $\varphi(1)(x) = ax$ .
3. If  $2|n$  and  $4 \nmid n$  then

$$D_{2n} \simeq D_n \times \mathbb{Z}_2.$$

4. There are no other nontrivial external semidirect decompositions of  $D_{2n}$  in the following sense. If  $D_{2n} \simeq X \rtimes Y$ ,  $|X| > 1$ ,  $|Y| > 1$ , then there are two possibilities:

- a)  $X = \mathbb{Z}_m$  and  $Y = D_{\frac{2n}{m}}$ , where  $m|n$ ,  $GCD(m, \frac{n}{m}) = 1$  or
- b)  $X = D_n$  and  $Y = \mathbb{Z}_2$ , if  $2|n$ .

*Proof.* Statements 1.,2.,3. are proved by exhibiting a suitable internal semidirect decomposition.

1. We use the primary decomposition theorem for cyclic groups: if  $n = \prod_{i=1}^k p_i^{\alpha_i}$ , then  $\mathbb{Z}_n \simeq \bigoplus_{i=1}^k \mathbb{Z}_{p_i^{\alpha_i}}$ . The statement follows from Proposition 2.4. Note that  $\text{Ker}(\varphi) = \langle a \rangle$ .

Alternatively, we prove the same statement using the information about  $D_{2n}$ -subgroups. We show that if  $GCD(m, \frac{n}{m}) = 1$  then  $D_{2n} = A_m \rtimes B_{\frac{2n}{m},r}$ .

We have that  $A_m \trianglelefteq D_{2n}$  and  $|A_m| \cdot |B_{\frac{2n}{m},r}| = 2n = |D_{2n}|$ .  $A_m \cap B_{\frac{2n}{m},r} \leq \langle a^{\frac{n}{m}} \rangle$ . Considering subgroups of  $\langle a^{\frac{n}{m}} \rangle$  it follows from Lemma 2.14 that  $A_m \cap B_{\frac{2n}{m},r} = \{e\}$ .



Thus  $D_{2n} = A_m \rtimes B_{\frac{2n}{m},r} \simeq_{\varphi} \mathbb{Z}_m \rtimes_{\varphi} D_{\frac{2n}{m}}$ . A direct computation shows that  $\varphi$  is as stated:  $(a^m)a^d(a^{-m}) = a^d$ ,  $(a^r x)a^d(a^r x) = a^{-d}$ .

Note that if  $2|n$  and  $4 \nmid n$  then  $A_2 \cap B_{n,r} = \{e\}$ ,  $r \in \mathbb{Z}_2$ , hence  $D_{2n} = A_2 \times B_{n,r} \simeq \mathbb{Z}_2 \times D_n$ . In this case there are no nontrivial semidirect decompositions of type  $\mathbb{Z}_2 \rtimes D_n$ .

2. This case is not covered by Proposition 2.4, we show directly that  $D_{2n} = B_{n,0} \rtimes B_{2,1}$ .

If  $2|n$ , then  $B_{n,0} \leq D_{2n}$ .  $|B_{n,0}| \cdot |B_{2,1}| = |D_{2n}|$ . It can be checked that  $B_{n,0} \cap B_{2,1} = \{e\}$ :  $B_{n,0} = \langle a^2, x \rangle$ ,  $B_{2,1} = \langle ax \rangle$ .

Thus  $D_{2n} \simeq D_n \rtimes \mathbb{Z}_2$ . A direct computation shows that  $\varphi$  is as stated:  $(ax)a^2(ax) = a^{-2}$  (the generator  $a^2$  gets inverted),  $(ax)x(ax) = a^2x$  (the generator  $x$  gets multiplied by the other generator  $a^2$ ).

3. Using Proposition 2.4 we see that  $D_{2n} = D(\mathbb{Z}_n) = (\mathbb{Z}_2 \oplus \dots) \rtimes \mathbb{Z}_2 \simeq \mathbb{Z}_2 \times D(\mathbb{Z}_n/\mathbb{Z}_2) \simeq \mathbb{Z}_2 \times D_n$ .

It can also be proved using the list of subgroups. We remind that  $D_{2n} = B_{n,0} \times A_2 \simeq D_n \times \mathbb{Z}_2$  for the following reasons. Both subgroups are normal.  $|B_{n,0}| \cdot |A_2| = |D_{2n}|$ .  $B_{n,0} = \langle a^2, x \rangle$ ,  $A_2 = \langle a^{\frac{n}{2}} \rangle$ ,  $\frac{n}{2}$  is odd, therefore  $B_{n,0} \cap A_2 = \{e\}$ .

4. Consider all possible internal semidirect decompositions of  $D_{2n}$ .

If  $D_{2n} = X \rtimes Y$  then  $X$  must be a normal subgroup of  $D_{2n}$  therefore  $X$  must be  $A_m$  or  $B_{n,r}$  with  $2|n$ .

If  $X = A_m$  then  $Y$  must be  $B_{m',r}$  in order to generate  $D_{2n}$ , with  $m' = \frac{2n}{m}$ .  $A_m \cap B_{\frac{2n}{m},r} = \{e\}$  iff  $GCD(n, \frac{n}{m}) = 1$  by Lemma 2.14.

Let  $X = B_{n,r}$  with  $2|n$ ,  $r \in \mathbb{Z}_2$ . There are  $n+1$  subgroups of  $D_{2n}$  having order 2:  $B_{2,r}$ ,  $r \in \mathbb{Z}_n$  and  $A_2 = \langle a^{\frac{n}{2}} \rangle$ . For any  $n$  such that  $2|n$  this gives a semidirect decomposition of type  $D_n \rtimes \mathbb{Z}_2$ . If  $4 \nmid n$  then  $A_2 \cap B_{n,r} = \{e\}$  which gives a direct decomposition  $D_n \times \mathbb{Z}_2$ .  $\square$

LEMMA 2.14. *Let  $n, a, b \in \mathbb{N}$ ,  $a|n$ ,  $b|n$ . Then*

$$|\mathbb{Z}_a \cap \mathbb{Z}_b| > 1 \text{ iff } GCD(a, b) > 1.$$

*Proof.* As it was noted above, if  $m|n$  then we identify  $\mathbb{Z}_m$  with the appropriate subgroup of  $\mathbb{Z}_n$ . Direct computations show that  $GCD(a, b) = d$  iff  $LCM(\frac{n}{a}, \frac{n}{b}) = \frac{n}{d}$ .

Using the multiplicative notation of cyclic groups we can assume that  $\mathbb{Z}_n = \langle z | z^n = e \rangle$ ,  $\mathbb{Z}_a = \langle z^{\frac{n}{a}} \rangle$ ,  $\mathbb{Z}_b = \langle z^{\frac{n}{b}} \rangle$ .  $|\mathbb{Z}_a \cap \mathbb{Z}_b| > 1$  iff  $LCM(\frac{n}{a}, \frac{n}{b}) < n$  iff  $GCD(a, b) > 1$ .  $\square$

REMARK 2.15. *In terms of prime factorization the condition  $GCD(m, \frac{n}{m}) = 1$  is equivalent to the fact that  $m$  and  $\frac{n}{m}$  are products of full prime powers of the prime factorization of  $n$ . Existence of many members of this family also follows from Schur-Zassenhaus theorem. If  $m|n$  and  $GCD(m, \frac{n}{m}) = 1$  then  $GCD(|A_m|, |D_{2n}/A_m|) = 1$ ,  $D_{2n}/A_m \simeq D_{\frac{2n}{m}}$  and, hence  $D_{2n} \simeq A_m \rtimes D_{\frac{2n}{m}}$ .*

REMARK 2.16. *Note that there are at most 2 external semidirect decompositions when  $n$  is a prime power:*

1. *if  $n = p^\alpha$ ,  $p$  an odd prime, then there is only one (classical) external semidirect decomposition:  $D_{2p^\alpha} \simeq \mathbb{Z}_{p^\alpha} \rtimes \mathbb{Z}_2$ ,*
2. *if  $n = 2^\alpha$ ,  $\alpha \geq 3$ , then there are two external semidirect decompositions:  $D_{2 \cdot 2^\alpha} \simeq \mathbb{Z}_{2^\alpha} \rtimes \mathbb{Z}_2 \simeq D_{2^\alpha} \rtimes \mathbb{Z}_2$ .*

REMARK 2.17. *The image of the twisting homomorphism in each case of a proper semidirect product is isomorphic to  $\mathbb{Z}_2$ . If the extending group is not  $\mathbb{Z}_2$ , then the twisting homomorphism is not injective.*

EXAMPLE 2.18.

*External semidirect decompositions of  $D_{2 \cdot 30}$ :*

$$\begin{aligned} D_{60} &\simeq \mathbb{Z}_{30} \rtimes \mathbb{Z}_2 \simeq \mathbb{Z}_6 \times D_{10} \simeq \mathbb{Z}_{10} \rtimes D_6 \simeq \mathbb{Z}_{15} \rtimes D_4 \simeq \mathbb{Z}_3 \times D_{20} \simeq \mathbb{Z}_5 \times D_{12} \simeq \\ &\simeq D_{30} \rtimes \mathbb{Z}_2 \simeq D_{30} \times \mathbb{Z}_2. \end{aligned}$$

*Internal semidirect decompositions of  $D_{2n}$ .* We now describe all internal semidirect decompositions of  $D_{2n}$ .

PROPOSITION 2.19. *Let  $n \in \mathbb{N}$ .*

1. *If  $m \in \mathbb{N}$ ,  $m|n$ , is such that  $GCD(m, \frac{n}{m}) = 1$ , then*

$$D_{2n} = A_m \rtimes B_{\frac{2n}{m}, r},$$

*for all  $r \in \mathbb{Z}_m$ .*

2. *If  $n = 2^\alpha q$ ,  $\alpha \in \mathbb{N}$ , then*

$$D_{2n} = B_{n,0} \rtimes B_{2,r_1} = B_{n,1} \rtimes B_{2,r_0}$$

*where  $r_i \in \mathbb{Z}_n$ ,  $r_i \equiv i \pmod{2}$ .*

3. *If  $2|n$  and  $4 \nmid n$  then  $D_{2n} = B_{n,0} \times A_2$  and  $D_{2n} = B_{n,1} \times A_2$ .*
4. *There are no other internal semidirect decompositions of  $D_{2n}$ .*

*Proof.*

1. We look for internal semidirect decompositions of  $D_{2n}$  in form  $A_m \rtimes B_{m',r}$ . We must have  $m' = \frac{2n}{m}$  and  $r \in \mathbb{Z}_m$ .  $A_m \cap B_{\frac{2n}{m},r} = \{e\}$  iff  $GCD(m, \frac{n}{m}) = 1$ . Thus

$D_{2n} = A_m \rtimes B_{\frac{2n}{m}, r}$  for all  $m$  such that  $GCD(m, \frac{n}{m}) = 1$  and all  $r \in \mathbb{Z}_m$  are the only possible decompositions of this kind.

2. We look for internal semidirect decompositions of  $D_{2n}$  in form  $B_{m,r} \rtimes B_{m',r'}$ . We must have  $B_{m,r} \trianglelefteq D_{2n}$  therefore  $2|n$ ,  $m = n$  and  $r \in \mathbb{Z}_2$ , thus we have two possible decomposition series:  $B_{n,0} \rtimes B_{2,r'}$  and  $B_{n,1} \rtimes B_{2,r''}$ . To ensure trivial intersections of semidirect factors we must have  $r' \equiv 1 \pmod{2}$  and  $r'' \equiv 0 \pmod{2}$ .

3. If  $2|n$  and  $4 \nmid n$  then  $B_{n,0} \cap A_2 = B_{n,1} \cap A_2 = \{e\}$  where all subgroups are normal.

4. It follows from the previous arguments.  $\square$

**3. Conclusion.** We have obtained results showing possibility of various semidirect decompositions of a given semidirect product in two cases: 1) if the original twisting homomorphism is diagonal and 2) if the extending group is directly decomposable. These results may stimulate further interest in developing Krull-Remak-Schmidt theorem type results for semidirect and Zappa-Szep products.

As an application we have presented semidirect decompositions of generalized dihedral groups and all possible external and internal semidirect decompositions of classical dihedral groups. Apart from semidirect decompositions guaranteed by the general proposition, for  $D_{2n}$  there are additional decompositions of external type  $D_n \rtimes \mathbb{Z}_2$  if  $2|n$ .

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