A NOTE ON ANOTHER CONSTRUCTION OF GRAPHS WITH $2^N + 6$ VERTICES AND CYCLIC AUTOMORPHISM GROUP OF ORDER 2^N

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Abstract. The problem of finding upper bounds for minimal vertex number of graphs with a given automorphism group is addressed in this article for the case of cyclic 2-groups. This problem was considered earlier by other authors. We give a construction of an undirected graph having $2^n + 6$ vertices and automorphism group cyclic of order 2^n, $n \geq 2$. This can revive interest in related problems.

Key words. graph, automorphism group

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1. Introduction.

This article addresses a problem in graph representation theory of finite groups - finding undirected graphs with a given full automorphism group and minimal number of vertices. All graphs in this article are undirected and simple.

It is known that finite graphs universally represent finite groups: for any finite group G there is a finite graph $\Gamma = (V, E)$ such that $Aut(\Gamma) \simeq G$, see Frucht [8]. It was proved by Babai [2] constructively that for any finite group G (except cyclic groups of order $3, 4$ or 5) there is a graph Γ such that $Aut(\Gamma) \simeq G$ and $|V(\Gamma)| \leq 2|G|$ (there are 2 G-orbits having $|G|$ vertices each). For certain group types such as symmetric groups Σ_n, dihedral groups D_n and elementary abelian 2-groups $(\mathbb{Z}/2\mathbb{Z})^n$ graphs with smaller number of vertices (respectively, n, n and $2n$) are obvious.

In the recent decades the problem of finding $\mu(G) = \min\limits_{\Gamma : Aut(\Gamma) \simeq G} |V(\Gamma)|$ for specific groups G does not seem to have been very popular although minimal graphs and directed graphs for most finite groups have not been found. See Babai [3] for an exposition of this area.

For $\mathbb{Z}/4\mathbb{Z}$ Babai's bound for vertices is not sharp. There are 10-vertex graphs having automorphism group $\mathbb{Z}/4\mathbb{Z}$, this fact is mentioned in Bouwer and Frucht [5] and Babai [2]. There are 12 such 10-vertex graph isomorphism types, see [6].

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In this paper we reminisce about the bound $\mu(G) = \min_{\Gamma: Aut(\Gamma)\simeq G} |V(\Gamma)| \leq 2|G|$ not being sharp for $G \simeq \mathbb{Z}/2^n\mathbb{Z}$, for any natural $n \geq 2$. Namely, for any $n \geq 2$ there is an undirected graph Γ on $2^n + 6$ vertices such that $Aut(\Gamma) \simeq \mathbb{Z}/2^n\mathbb{Z}$. The number of orbits is 3. In this paper we are not concerned with minimization of number of edges.

Graphs with abelian automorphism groups have been investigated in Arlinghaus [1]. In Harary [9] there is an exercise claiming (referring to Merriwether) that if G is a cyclic group of order 2^n, $n \geq 2$, then the minimal number of graph vertices is $2^n + 6$. In this paper we exhibit such graphs and give an explicit construction.

We use standard notations of graph theory, see Diestel [7]. Adjacency of vertices i and j is denoted by $i \sim j$, an undirected edge between $i \sim j$ is denoted by (i, j). For a graph $\Gamma = (V, E)$ the subgraph induced by $X \subseteq V$ is denoted by $\Gamma[X]$: $\Gamma[X] = \Gamma - \overline{X}$. The set $\{1, 2, ..., n\}$ is denoted by V_n, we assume that $V_m \subseteq V_n$ if $m \leq n$. The undirected cycle on n vertices is denoted by C_n. The cycle notation is used for permutations. Given a function $f : A \mapsto B$ and a subset $C \subseteq A$ we denote the restriction of f to C by $f|_C$.

2. Main results.

2.1. The graph Γ_n.

Definition 2.1. *Let $n \geq 2$, $n \in \mathbb{N}$, $m = 2^n$. Let $V(\Gamma_n) = V_{m+6} = \{1, 2, ..., m + 6\} = \{1, 2, ..., 2^n + 6\}$ and edges be given by the following adjacency description. We define 8 types of edges.*

1. *$i \sim i+1$ for all $i \in V_{m-1}$ and $1 \sim m$. (It indicates that $\Gamma_m[1, 2, ..., m] \simeq C_m$).*
2. *$m + 1 \sim i$ with $i \in V_m$ iff $i \equiv 1$ or $2 (mod\ 4)$.*
3. *$m + 2 \sim i$ with $i \in V_m$ iff $i \equiv 2$ or $3 (mod\ 4)$.*
4. *$m + 3 \sim i$ with $i \in V_m$ iff $i \equiv 3$ or $0 (mod\ 4)$.*
5. *$m + 4 \sim i$ with $i \in V_m$ iff $i \equiv 0$ or $1 (mod\ 4)$.*
6. *$m + 5 \sim i$ with $i \in V_m$ iff $i \equiv 1 (mod\ 2)$.*
7. *$m + 6 \sim i$ with $i \in V_m$ iff $i \equiv 0 (mod\ 2)$.*
8. *$m + 1 \sim m + 5 \sim m + 3$, $m + 2 \sim m + 6 \sim m + 4$.*

Definition 2.2. *Denote $O_1 = \{1, 2, ..., m\}$, $O_2 = \{m + 1, m + 2, m + 3, m + 4\}$, $O_3 = \{m + 5, m + 6\}$. Note that O_i are the $Aut(\Gamma_n)$-orbits.*

2.2. The special case $n = 2$.

A graph with automorphism group $\mathbb{Z}/4\mathbb{Z}$ and minimal number of vertices (10) and edges (18) was exhibited in Bouwer and Frucht [5], p.58. Γ_2 (which is not isomorphic to the Bouwer-Frucht graph) is shown in Fig.1. It can be thought as embedded in the

$3D$ space. It is planar but a plane embedding is not given here. $Aut(\Gamma_2) \simeq \mathbb{Z}/4\mathbb{Z}$ is generated by the vertex permutation $g = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10)$.

Subgraphs $\Gamma_2[1, 2, 3, 4, 5, 7, 9]$ and $\Gamma_2[1, 2, 3, 4, 6, 8, 10]$ which can be thought as being drawn above and below the orbit $\{1, 2, 3, 4\}$ are interchanged by g.

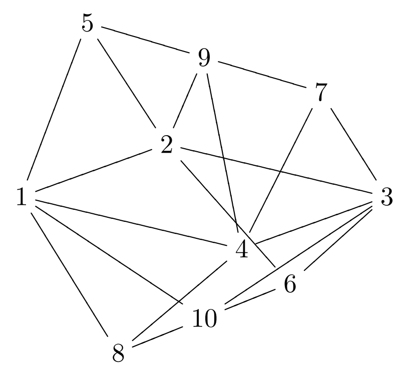

Fig.1. - Γ_2

2.3. Automorphism group of Γ_n.

PROPOSITION 2.3. *Let $n \geq 2$, $n \in \mathbb{N}$, $m = 2^n$. Let Γ_n be defined as above. For any n, $Aut(\Gamma_n) \simeq \mathbb{Z}/m\mathbb{Z}$.*

Proof. We will show that $Aut(\Gamma_n) = \langle g \rangle$, where $g = (1, 2, ..., m)(m+1, m+2, m+3, m+4)(m+5, m+6)$.

Inclusion $\langle g \rangle \leq Aut(\Gamma_n)$ is proved by showing that g maps an edge of each type to an edge.

Let us prove the inclusion $Aut(\Gamma_n) \leq \langle g \rangle$. Let $f \in Aut(\Gamma_n)$. We will show that $f = g^\alpha$ for some α. There are two subcases: $n \neq 3$ and $n = 3$.

For any $n \geq 2$ the vertices $m+5$ and $m+6$ are the only vertices having eccentricity 3, so they must be permuted by f.

Let $n \neq 3$. Suppose $f(1) = k$. Since $n \neq 3$, we have that $deg(1) = 5$, $deg(v) = \frac{m}{2}+1 \neq 5$ for any $v \in O_2$, therefore $f(1) \in O_1$. Moreover, f stabilizes setwise both O_1 and O_2. Consider the f-image of the edge $(1, m+5)$. $(f(1), f(m+5)) = (k, f(m+5))$ must be an edge, therefore

1. if $k \equiv 1(mod\ 2)$, then $f(m + 5) = m + 5$,
2. if $k \equiv 0(mod\ 2)$, then $f(m + 5) = m + 6$.

It follows that $f|_{O_3} = g^{k-1}$.

Consider the f-image of $\Gamma_n[1, 2, m+1, m+5]$, denote its isomorphism type by Γ_0, see Fig.5.

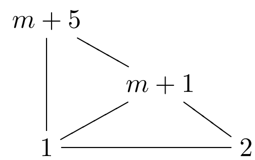

Fig.5. - $\Gamma_0 \simeq \Gamma_n[1, 2, m+1, m+5]$.

Vertex 2 must be mapped to a $\Gamma_n[O_1]$-neighbour of k. For any $k \in O_1$ there are two triangles containing the vertex k and a vertex adjacent to k in $\Gamma_n[O_1]$. Taking into account that $f(m+5) \in O_3$ we check that there is only one suitable induced Γ_n-subgraph - containing k, another vertex in O_1 adjacent to k and a vertex in O_3 - which is isomorphic to $\Gamma_n[1, 2, m+1, m+5]$. It follows that in each case we must have $f(2) \equiv k+1 (mod\ m)$. By similar arguments for all $j \in \{1, 2, .., m\}$ it is proved that $f(j) \equiv (k-1) + j (mod\ m)$, thus $f|_{O_1} = g^{k-1}$.

Finally we describe $f|_{O_2}$. It can also be found considering Γ_n-subgraphs isomorphic to Γ_0, but we will use edge inspection. Consider the f-images of the edges $(1, m+1)$ and $(1, m+4)$. Vertex pairs $(f(1), f(m+1)) = (k, f(m+1))$ and $(f(1), f(m+4))$ must be edges, therefore we can deduce images of all O_2 vertices. If $n \neq 3$ and $f(1) = k$, then $f = g^{k-1}$, therefore $f \in \langle g \rangle$.

In the special case $n = 3$ we also consider f-images of $\Gamma_3[1, 2, 9, 13]$ and find suitable Γ_3-subgraphs isomorphic to Γ_0. It is shown similarly to the above argument that f can be expressed as a power of g and hence $f \in \langle g \rangle$. $\square$

2.4. Abelian 2-groups. In this subsection we give an immediate implication - a bound for $\mu(G)$ if G is an abelian 2-group.

PROPOSITION 2.4. *Let G be an abelian 2-group: $G \simeq \prod_{i=1}^{k} (\mathbb{Z}/2^i\mathbb{Z})^{n_i}$, $n_i \in \mathbb{N} \cap \{0\}$. Then $\mu(G) \leq 2n_1 + \sum_{i=2}^{k} n_i(2^{n_i} + 6)$.*

Proof. Denote $(\mathbb{Z}/2^i\mathbb{Z})^{n_i}$ by G_i, $G \simeq \prod_{i=1}^{k} G_i$. We can construct a sequence of graphs $\Delta_{i,n}$, $i \in \mathbb{N}$, $n \in \mathbb{N}$, inductively using complements and unions as follows. For $i > 1$ define $\Delta_{i,1} = \Gamma_i$ and define $\Delta_{1,1} = K_2$. Define inductively $\Delta_{i,n}$: $\Delta_{i,n} = \overline{\Delta_{i,n-1}} \cup \Delta_{i,1}$. Since $\overline{\Delta_{i,n-1}} \not\simeq \Delta_{i,1}$ and $\overline{\Delta_{i,j}}$ is connected for all constructed graphs, we have inductively that $Aut(\Delta_{i,n}) \simeq Aut(\Delta_{i,n-1}) \times (\mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2^i\mathbb{Z})^n$.

Define $\Gamma = \bigcup_{i=1}^{k} \Delta_{i,n_i}$. For different values of i the Δ_{i,n_i} are nonisomorphic therefore $Aut(\Gamma) \simeq \prod_{i=1}^{k} G_i \simeq G$. Thus $\mu(G) \leq |V(\Gamma)| = \sum_{i=1}^{k} |V(\Delta_{i,n_i})| = 2n_1 + \sum_{i=2}^{k} n_i(2^{n_i} + 6)$. □

2.5. Other graphs and developments. We briefly describe without proofs graphs $\Gamma_{m,n}$ having $m^n + m$ vertices and cyclic automorphism group of order m^n, $m \geq 6$. Existence of such graphs is mentioned in [9]. We use the construction of graphs with $2m$ vertices having cyclic automorphism group of order m ($m \geq 6$) given in [11]. Let $V(\Gamma_{m,n}) = W \cup W'$, where $W = \{0, 1, ..., m^n-1\}$, $W' = \{0', 1' ..., (m-1)'\}$. The edges of $\Gamma_{m,n}$ are defined as follows: 1) $\Gamma_{m,n}[W]$ and $\Gamma_{m,n}[W']$ are natural cycles of order m^n and m, respectively, with edges $(i, i+1)$, 2) for any vertex $i' \in W'$ there are $3m^{n-1}$ edges of type $(i', jm + i(mod\ m^n))$, $(i', jm + i + 1(mod\ m^n))$ and $(i', jm + i - 2(mod\ m^n))$, $0 \leq i' \leq m-1$, $0 \leq j \leq m^{n-1} - 1$. It can be checked that $Aut(\Gamma_{m,n}) \simeq \mathbb{Z}/m^n\mathbb{Z}$, there are 2 orbits - W and W'.

2.6. Conclusion. For a finite group G Babai's construction requires $2|G|$ vertices for a graph to have automorphism group isomorphic to G, two orbits having $|G|$ elements each. Except for some small groups and series such as symmetric groups the exact minimal number of vertices for undirected (and directed) graphs remains an unsolved and unpopular problem which should be revived. We have given an explicit and relatively easy construction of graphs with $2^n + 6$ vertices and the automorphism group G cyclic of order 2^n.

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